Thesis Proposal: Encoding Invariance of Optimal Approximate Quantum Cloning Fidelity

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1 Abstract

The no-cloning theorem [1] forbids perfect duplication of arbitrary unknown quantum states, yet approximate cloning is permitted with optimal fidelity bounds derived through quantum channel optimization. These bounds are traditionally formulated for physical qubits over the full Hilbert space, using symmetry arguments that pin the universal $1 \to 2$ cloning fidelity to $5/6$ [2]. In practical quantum information processing, however, information is encoded into logical qubits using stabilizer codes, where states reside in protected subspaces of larger physical systems. Under such encodings the relevant symmetry group is reduced from the full $SU(2)$ to a conjugated subgroup acting only on the two-dimensional code subspace, weakening the covariance constraint that fixes the $5/6$ bound and potentially admitting higher fidelity.

This thesis investigates whether optimal approximate cloning fidelity is invariant under stabilizer encoding of logical qubits, or whether the restriction to code subspaces modifies the achievable bound. The problem is formulated as a semidefinite programme over completely positive trace-preserving (CPTP) maps acting on encoded states, with fidelity evaluated over the logical Bloch sphere via the Choi–Jamiołkowski isomorphism. Analytical investigation combines symmetry and covariance arguments with stabilizer formalism; numerical results are obtained by solving the SDP for small code instances including the $[[5, 1, 3]]$ and $[[7, 1, 3]]$ codes. The analysis is extended to $1 \to M$ cloning, asymmetric cloning, and constrained fault-tolerant channels.

The results have direct implications for quantum cloud storage, quantum key distribution security, and quantum repeater design, areas in which logical qubits are the operational unit and the achievable cloning fidelity governs fundamental performance limits. The work contributes to the theoretical foundations of encoded quantum information processing and connects to the active contemporary research programmes on encrypted cloning [3] and quantum channel cloning [4].

2 Problem Statement

Let $\mathcal{H}_L$ denote the logical Hilbert space of a single qubit and $\mathcal{H}_P$ the physical Hilbert space associated with an n-qubit stabilizer code. Let

$$V : \mathcal{H}_L \to \mathcal{H}_P$$

be an isometric encoding map whose image defines the code subspace $\mathcal{C} \subseteq \mathcal{H}_P$.

Consider a completely positive trace-preserving (CPTP) map

$$\mathcal{E} : \mathcal{B}(\mathcal{H}_P) \to \mathcal{B}(\mathcal{H}_P \otimes \mathcal{H}_P),$$

which is intended to implement approximate $1 \to 2$ cloning of encoded logical states. For an input logical state $|\psi\rangle \in \mathcal{H}_L$, the encoded state is $V|\psi\rangle \in \mathcal{C}$, and the output of the cloning channel is given by $\mathcal{E}(V|\psi\rangle\langle\psi|V^\dagger)$.

We define the logical cloning fidelity as the average state fidelity between the ideal duplicated encoded state and the actual output of the channel:

$$F = \int d\psi \, F\Big(\mathcal{E}(V|\psi\rangle\langle\psi|V^\dagger), (V|\psi\rangle\langle\psi|V^\dagger)^{\otimes 2}\Big),$$

where the integral is taken over the uniform (Haar) measure on the Bloch sphere.

The central problem of this thesis is to characterize the optimal achievable fidelity over all CPTP maps $\mathcal{E}$ acting on the physical Hilbert space, under the

restriction that inputs lie within the encoded code subspace.

Research Question. Does the optimal approximate cloning fidelity for logical qubits remain invariant under isometric stabilizer encoding, or does restriction to stabilizer code subspaces modify the fundamental fidelity bounds known for physical qubits?

Equivalently, the problem seeks to determine whether the optimal fidelity for approximate cloning depends only on the abstract logical Hilbert space structure, or whether additional constraints imposed by stabilizer encoding—such as code distance, degeneracy, and subspace geometry—lead to deviations from known universal cloning limits.

2.1 Mathematical Motivation for the Research Question

The central research question asks whether the optimal approximate cloning fidelity for logical qubits, as defined in Section 2, equals or deviates from the known universal bound of 5/6 for physical qubits [2]. Rather than accepting this as a plausible conjecture, this section derives the 5/6 bound from first principles, identifies the structural assumption on which it rests, and shows explicitly that stabilizer encoding invalidates that assumption. This provides a rigorous mathematical basis for the investigation.

The 5/6 bound and its symmetry origin. The Bužek–Hillery result follows from a covariance argument over the qubit state space [2]. A cloning channel $\mathcal{E}_{\text{phys}} : \mathcal{B}(\mathcal{H}_L) \to \mathcal{B}(\mathcal{H}_L^{\otimes 2})$ is called *universal* if its performance is state-independent, i.e., the cloning fidelity is the same for every input $|\psi\rangle \in \mathcal{H}_L$. This is equivalent to requiring that $\mathcal{E}_{\text{phys}}$ be covariant with respect to the full $SU(2)$ symmetry group acting on $\mathcal{H}_L \cong \mathbb{C}^2$:

$$\mathcal{E}_{\text{phys}}(U\rho U^\dagger) = (U \otimes U)\mathcal{E}_{\text{phys}}(\rho)(U^\dagger \otimes U^\dagger) \quad \forall U \in SU(2). \quad (1)$$

By Schur's lemma applied to the tensor decomposition of $SU(2)$ representations, any channel satisfying (1) must have a Choi matrix $J(\mathcal{E}_{\text{phys}})$ of the specific form

$$J(\mathcal{E}_{\text{phys}}) = \alpha\Pi_{\text{sym}} + \beta\Pi_{\text{asym}}, \quad (2)$$

where Π_{sym} and Π_{asym} are the projectors onto the symmetric and antisymmetric subspaces of $\mathcal{H}_L^{\otimes 2}$, and α, β are constrained by the CPTP conditions. Maximizing the average fidelity over all channels of this form yields the unique optimal cloning channel with fidelity

$$F^*_{\text{phys}} = \max_{\mathcal{E} \text{ s.t. } (1)} \int d\psi F(\mathcal{E}(|\psi\rangle\langle\psi|), |\psi\rangle\langle\psi|^{\otimes 2}) = \frac{5}{6}. \quad (3)$$

The key structural assumption underlying (3) is that covariance is required under the *full* $SU(2)$ group acting on $\mathcal{H}_L$, and that the input distribution is the uniform (Haar) measure over all pure qubit states. Both conditions are inherited from the physical problem, where no additional structure is imposed on the input space.

How stabilizer encoding modifies the symmetry constraint. Under the encoding map $V : \mathcal{H}_L \to \mathcal{H}_P$, the relevant symmetry group for logical cloning is no longer the full $SU(2)$ acting on $\mathcal{H}_P$. The physically meaningful symmetry is the image of $SU(2)$ under conjugation by V, namely the group

$$\mathcal{G}_L = \{VUV^\dagger : U \in SU(2)\} \subset \mathcal{U}(\mathcal{H}_P). \quad (4)$$

Since V is an isometry ($V^\dagger V = I_L$), the group $\mathcal{G}_L$ acts as $SU(2)$ on the two-dimensional code subspace $\mathcal{C} = V\mathcal{H}_L$, but acts as the *identity* on the orthogonal complement $\mathcal{C}^\perp \subset \mathcal{H}_P$. For an $[[n, 1, d]]$ stabilizer code, $\dim(\mathcal{H}_P) = 2^n$ while $\dim(\mathcal{C}) = 2$, so $\mathcal{C}^\perp$ has dimension $2^n - 2$.

The covariance condition for the logical cloning channel $\mathcal{E} : \mathcal{B}(\mathcal{H}_P) \to \mathcal{B}(\mathcal{H}_P^{\otimes 2})$ is therefore

$$\mathcal{E}(VUV^\dagger \rho VU^\dagger V^\dagger) = (VU \otimes VU)\mathcal{E}(\rho)(VU^\dagger \otimes VU^\dagger) \quad \forall U \in SU(2). \quad (5)$$

which is a covariance constraint under $\mathcal{G}_L$, not under the full unitary group $SU(2^n)$ on $\mathcal{H}_P$. Since $\mathcal{G}_L$ acts trivially on $\mathcal{C}^\perp$, constraint (5) places *no symmetry restriction* on how $\mathcal{E}$ acts on states supported in $\mathcal{C}^\perp$. In the physical problem, the full $SU(2)$ covariance on $\mathcal{H}_L$ fixes the Choi matrix of the optimal channel to the form (2), leaving only two free parameters. In the logical problem, the weaker constraint (5) admits a much larger class of Choi matrices, since the $(2^n - 2)$-dimensional complement $\mathcal{C}^\perp$ is unconstrained.

This establishes the following formal observation.

Remark 1 (Relaxed Covariance Under Encoding)**.** *The feasible set of CPTP maps $\mathcal{E}$ satisfying the logical cloning covariance condition (5) strictly contains the*

feasible set satisfying the physical cloning covariance condition (1). Consequently, $F^*_{\text{logical}} \geq F^*_{\text{phys}} = 5/6$ holds as a lower bound, with the inequality potentially strict for sufficiently large n.

Remark 1 shows that the 5/6 bound is not a universal ceiling for logical cloning: it applies only under the full Bloch-sphere covariance assumption, which encoding breaks. The question of whether the bound is achieved, matched, or exceeded is precisely what this thesis investigates.

Precedent from phase-covariant cloning. An established result in the literature provides direct mathematical precedent for the idea that restricting the symmetry group can *increase* the achievable cloning fidelity. Bruss et al. [5] showed that when the input is restricted to *equatorial* states of the Bloch sphere, $|\psi_\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle)$, the relevant symmetry is reduced from $SU(2)$ to $U(1)$, and the optimal cloning fidelity improves to

$$F^*_{\text{phase}} = \frac{1}{2} + \frac{1}{2\sqrt{2}} \approx 0.854 > \frac{5}{6} \approx 0.833. \tag{6}$$

The improvement arises because the reduced symmetry group $U(1) \subset SU(2)$ admits a larger class of covariant channels, allowing optimization over a strictly larger feasible set.

The structural analogy to the encoded cloning problem is precise. In phase-covariant cloning, the input symmetry is reduced from $SU(2)$ to $U(1)$ by restricting to a one-dimensional orbit in state space. In logical cloning under stabilizer encoding, the physical covariance group is reduced from the full $SU(2^n)$ on $\mathcal{H}_P$ to the two-dimensional subgroup $\mathcal{G}_L \cong SU(2)$ by restricting the input domain to the code subspace $\mathcal{C}$. In both cases, the effective symmetry constraint on the cloning channel is weakened, and optimization proceeds over a larger feasible set.

Conjecture and verification strategy. The foregoing analysis motivates the following conjecture, which constitutes the primary object of investigation in this thesis.

Conjecture 1 (Encoding-Enhanced Cloning Fidelity)**.** *For an $[[n, 1, d]]$ stabilizer code with $n \geq 3$ and encoding map $V : \mathcal{H}_L \to \mathcal{H}_P$, the optimal logical cloning fidelity satisfies*

$$F^*_{\text{logical}} = \sup_{\mathcal{E} \in \text{CPTP}} \int d\psi \, F(\mathcal{E}(V|\psi\rangle\langle\psi|V^\dagger), (V|\psi\rangle\langle\psi|V^\dagger)^{\otimes 2}) \geq \frac{5}{6}. \tag{7}$$

with the inequality potentially strict for codes whose stabilizer structure imposes a non-trivial geometry on the code subspace $\mathcal{C}$. Whether strict inequality holds, or whether $F^*_{\text{logical}} = 5/6$ for all stabilizer codes, is the central question of this thesis.

The conjecture is empirically testable via semidefinite programming. Using the Choi–Jamiołkowski isomorphism, the optimal fidelity (7) is the value of the following SDP:

$$\begin{aligned} \underset{J \in \mathcal{B}(\mathcal{H}_P^{\otimes 3})}{\text{maximize}} \quad & \text{Tr}[M_V J] \\ \text{subject to} \quad & J \geq 0, \\ & \text{Tr}_{\text{out}} J = I_{\mathcal{H}_P}, \end{aligned} \tag{8}$$

where J is the Choi matrix of $\mathcal{E}$, Tr_{out} denotes the partial trace over the two output subsystems, and M_V is the objective operator encoding the average fidelity over the logical Bloch sphere under encoding V. For the $[[5, 1, 3]]$ perfect quantum code, $\mathcal{H}_P \cong \mathbb{C}^{32}$, and the SDP (8) involves a $32^3 = 32768$-dimensional variable, which is tractable with standard solvers. The SDP will produce a definitive numerical answer: $F^*_{\text{logical}} = 5/6$ (encoding invariance) or $F^*_{\text{logical}} > 5/6$ (encoding-enhanced fidelity), either of which constitutes a meaningful theoretical contribution to the characterisation of quantum information limits under stabilizer encoding.

3 Background and Literature Review

3.1 No-Cloning Theorem

The no-cloning theorem establishes that it is impossible to construct a physical operation that perfectly duplicates an arbitrary unknown quantum state [1]. Formally, there exists no unitary or completely positive trace-preserving (CPTP) map that can implement the transformation $|\psi\rangle \mapsto |\psi\rangle \otimes |\psi\rangle$ for all states $|\psi\rangle$ in a continuous Hilbert space. This fundamental limitation arises from the linearity of quantum mechanics and plays a central role in quantum information theory, particularly in areas such as quantum cryptography and communication.

3.2 Approximate Quantum Cloning

While perfect cloning is forbidden, approximate cloning is permitted, where imperfect copies of a quantum state are produced with maximal achievable fidelity. The seminal work of Bužek and Hillery introduced the universal quantum cloning machine (UQCM), which achieves an optimal fidelity of 5/6 for symmetric $1 \to 2$ cloning of qubits [2]. This result is derived using symmetry and covariance arguments over the Bloch sphere and represents a fundamental bound for universal cloning. The construction was subsequently extended to the universal optimal cloning of qubit registers, establishing that covariance under the full unitary group uniquely determines the optimal channel [6].

The general problem of $N \to M$ cloning — producing $M > N$ approximate copies from N identical inputs — was solved independently by Gisin and Massar [7] and by Werner. The optimal fidelity for symmetric universal $1 \to M$ cloning of qubits is

$$F(M) = \frac{2M+1}{3M}, \tag{9}$$

which decreases monotonically from 5/6 at $M = 2$ toward 2/3 as $M \to \infty$ [7]. Werner's channel-theoretic derivation further established a connection between optimal cloning and optimal state estimation, showing that the Choi matrix of the optimal cloning channel has a specific representation-theoretic form determined by the symmetry of the input ensemble.

Subsequent work has explored state-dependent cloning scenarios, such as phase-covariant cloning, where higher fidelities can be achieved when the set of input states is restricted [5]. More recent studies have investigated applications of approximate cloning in quantum communication and information distribution, including multi-party transmission protocols and imperfect cloning-based extensions of communication capabilities [8, 9].

The scope of cloning has recently been extended beyond quantum states to the cloning of quantum channels themselves. Sekatski et al. [4] established a unified framework in which CPTP maps are treated as the resource to be cloned, demonstrating that state cloning is a special case of the more general channel cloning problem. They construct an SDP algorithm for computing optimal channel cloning fidelities and identify conditions under which *super-replication* — producing copies at a quadratic rate in the number of input copies — is possible. In a related computational direction, Bansal et al. [10] have shown that cloning stabilizer states is informationally as hard as learning them, thereby establishing a formal connection between the theory of quantum cloning and the complexity of learning structured quantum states.

3.3 Encoding and Quantum Error Correction

In practical quantum information processing, information is typically stored not in individual physical qubits but in encoded logical qubits using quantum error-correcting codes. The stabilizer formalism, developed by Gottesman, provides the foundational framework in which logical states are embedded into higher-dimensional physical Hilbert spaces via isometric encodings, enabling protection against noise and decoherence. An important family of stabilizer codes — CSS codes — was introduced by Calderbank and Shor, who showed that codes with desirable distance properties can be constructed systematically from pairs of classical linear codes.

Encoding maps define code subspaces with nontrivial algebraic structure, characterised by stabilizer generators, logical operators, and code distance. The stabilizer group $\mathcal{S}$ of an $[[n, k, d]]$ code is an abelian subgroup of the n-qubit Pauli group; the code subspace $\mathcal{C}$ is the simultaneous $+1$ eigenspace of all elements of $\mathcal{S}$. Logical operators act non-trivially on $\mathcal{C}$ while commuting with $\mathcal{S}$, and the code distance d determines the number of physical errors the code can correct. These features constrain the set of physically realisable operations and may influence the behaviour of quantum channels acting on encoded states.

3.4 Cloning Under Encoding: Open Questions

Despite the extensive study of approximate cloning at the level of physical qubits and the well-developed theory of stabilizer codes, the interaction between these domains remains largely unexplored. In particular, it is not known whether optimal cloning fidelity is invariant under isometric encoding into stabilizer code subspaces.

Recent work has begun to explore the relationship between encoding and cloning-like behaviour in specific contexts. Yamaguchi and Kempf [3] showed that encrypted qubits can be perfectly cloned: any number of encrypted clones of a qubit can be produced through a unitary transformation, with each clone decryptable via a single-use key, in full agreement with the no-cloning theorem. This result opened a new paradigm of *encrypted cloning*, with direct applications to quantum cloud storage and distributed

quantum computation [11].

The encrypted cloning framework has since generated a substantial body of follow-up work. Ceara [12] solved the open problem posed by Yamaguchi and Kempf by generalising the protocol to arbitrary-dimensional quantum systems (qudits), establishing that encrypted cloning is intrinsic to all finite-dimensional quantum information. The relationship between encrypted cloning and quantum secret sharing has been formalised by Lim and Lo [13], who showed that the entanglement structure required for encrypted cloning is equivalent to absolute maximal entanglement as used in quantum secret sharing protocols. Gianini et al. [14] extended this connection by deriving encrypted cloning protocols from general secret-sharing access structures, moving beyond the canonical Pauli-operator-based construction.

Simultaneously, foundational work on the limits of cloning has also advanced. Poremba et al. [15] introduced the framework of *cloning games* to capture quantitative versions of the no-cloning theorem and their cryptographic consequences, while Bhattacharyya, Broadbent, and Culf [16] proved the existence of the uncloneable bit with unconditional security, directly connecting approximate cloning bounds to cryptographic security levels. These developments demonstrate that cloning is not a settled topic but an active research frontier with significant implications for quantum cryptography, communication, and computation.

However, no general framework currently exists to analyse approximate cloning of logical qubits within stabilizer codes or to determine whether encoding modifies fundamental fidelity bounds. This gap motivates the present work, which seeks to establish whether approximate cloning fidelity is an encoding-invariant quantity or whether stabilizer structure induces systematic deviations.

3.5 Quantum Channel Theory and Semidefinite Programming

The analysis of optimal approximate cloning reduces naturally to an optimisation problem over CPTP maps. The central mathematical tool that makes this tractable is the Choi–Jamiołkowski (CJ) isomorphism, which establishes a bijection between CPTP maps and positive semidefinite matrices satisfying a linear trace constraint.

Specifically, for any CPTP map $\mathcal{E} : \mathcal{B}(\mathcal{H}_{\text{in}}) \to \mathcal{B}(\mathcal{H}_{\text{out}})$, the associated *Choi matrix* is defined as

$$J(\mathcal{E}) = (I \otimes \mathcal{E})|\Phi^+\rangle\langle\Phi^+| \in \mathcal{B}(\mathcal{H}_{\text{in}} \otimes \mathcal{H}_{\text{out}}), \quad (10)$$

where $|\Phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle$ is the maximally entangled state on $\mathcal{H}_{\text{in}} \otimes \mathcal{H}_{\text{in}}$. The map $\mathcal{E}$ is completely positive if and only if $J(\mathcal{E}) \geq 0$, and it is trace-preserving if and only if $\text{Tr}_{\text{out}}\, J(\mathcal{E}) = I_{\text{in}}$. These two conditions are affine constraints on $J(\mathcal{E})$, and the set of all CPTP maps is therefore a convex set in the space of matrices.

The average cloning fidelity, as defined in Section 2, is a linear functional of $J(\mathcal{E})$. This is because the average over the Haar measure of any polynomial expression in the channel output reduces, via the CJ isomorphism, to a linear trace of $J(\mathcal{E})$ against a fixed objective matrix M_V determined by the encoding V. Consequently, the problem of finding the optimal cloning channel $\mathcal{E}^*$ is equivalent to solving the semidefinite programme (SDP):

$$\begin{aligned} \underset{J}{\text{maximize}} \quad & \text{Tr}[M_V \cdot J] \\ \text{subject to} \quad & J \geq 0, \\ & \text{Tr}_{\text{out}}\, J = I_{\mathcal{H}_P}. \end{aligned} \quad (11)$$

This SDP is convex with a linear objective and affine constraints; it has a unique optimal solution and can be solved efficiently by standard interior-point solvers such as CVX or Mosek. For the $[[5, 1, 3]]$ perfect quantum code with $\dim(\mathcal{H}_P) = 2^5 = 32$, the variable J is a 32×32^2-dimensional positive semidefinite matrix, placing the problem well within the reach of current computational tools. The SDP-based approach has been used extensively in the quantum cloning literature to derive exact optimal fidelities for state-dependent cloning scenarios and to validate analytical bounds [4]. Its application to the encoded logical cloning problem constitutes the numerical component of the present research.

3.6 Applications and Practical Relevance

The question of whether approximate cloning fidelity is modified by stabilizer encoding is not merely of foundational interest. The answer has direct quantitative consequences for several areas of quantum technology in which logical qubits are the operational unit.

Quantum cloud computing and distributed state storage. Yamaguchi and Kempf [3] proposed encrypted cloning as the basis for quantum cloud storage — a setting in which the same logical quantum state is held redundantly across multiple servers without violating the no-cloning theorem. This concept has been demonstrated experimentally on IBM Heron-R2 superconducting processors using up to

154 physical qubits [12], establishing that cloning-based protocols are viable on current hardware. In cloud architectures that encode logical qubits for error protection, the achievable distribution fidelity is bounded by F^*_{logical}; if encoding alters this bound, it directly changes the resource requirements for multi-server quantum storage.

Security of quantum key distribution under encoding. In standard security proofs for quantum key distribution (QKD) protocols such as BB84 and E91, the optimal eavesdropping attack is modelled as an approximate cloning channel that maximises information gain subject to the 5/6 fidelity bound for physical qubits. When the communicating parties use encoded logical qubits — as required in long-distance QKD relying on quantum repeaters — the relevant eavesdropping bound becomes F^*_{logical}. The recent proof that an uncloneable bit exists with unconditional security [16] makes this connection explicit: approximate cloning bounds directly determine the secrecy thresholds of quantum cryptographic protocols, and those thresholds must be re-evaluated when encoding is present.

Quantum repeaters and one-to-many network routing. In quantum communication networks, repeater nodes must distribute quantum information to multiple downstream receivers. Approximate cloning is the information-theoretically optimal strategy for this one-to-many routing task, and the achievable fidelity per copy determines the end-to-end network performance [8]. Next-generation repeater architectures store and process quantum states as encoded logical qubits; the optimal logical cloning fidelity determines the maximum efficiency of cloning-based repeater nodes, a quantity that has not previously been characterised.

Quantum secret sharing. Quantum secret sharing protocols distribute a quantum secret among N parties such that any authorised subset can jointly recover it. These protocols are structurally related to cloning in that the dealer effectively produces N imperfect copies of the secret state. When the secret is encoded in a quantum error-correcting code for robustness, the optimal fidelity achievable by any distributor is bounded by F^*_{logical}. The formalisation of the relationship between encrypted cloning and quantum secret sharing [13, 14] makes this connection precise: the access structure of the secret-sharing scheme determines which subsets of clones can be jointly decrypted, and the encoding geometry directly governs the achievable fidelity of each share.

Benchmarking fault-tolerant quantum processors. As quantum processors move toward fault-tolerant operation, the relevant performance metric shifts from physical gate fidelity to logical operation fidelity. The optimal cloning fidelity for logical qubits provides a theoretical reference value against which experimental logical-level state distribution can be assessed. If encoding invariance holds, existing physical-qubit benchmarks transfer directly to the logical level; if the fidelity differs, a new class of logical-level benchmarks is required.

In each of these settings, the central quantity of practical interest is the optimal fidelity achievable by approximate cloning of encoded logical qubits — precisely the quantity this thesis aims to characterise.

4 Research Objectives

The primary objective of this thesis is to rigorously investigate whether optimal approximate quantum cloning fidelity is invariant under stabilizer encoding of logical qubits, and, in light of the relaxed symmetry constraint identified in Section 2.1, whether encoding can enable fidelity strictly exceeding the universal bound of 5/6. To address this central question, the work will pursue the following specific objectives:

- To formulate approximate cloning of logical qubits as an optimization problem over completely positive trace-preserving (CPTP) maps acting on encoded subspaces.
- To determine whether the optimal fidelity for $1 \to 2$ cloning of logical qubits coincides with, or exceeds, the known optimal bound of 5/6 for physical qubits [2], and to characterise the role of the code subspace geometry in either outcome.
- To investigate whether the $1 \to M$ generalisation of universal cloning, whose optimal physical-qubit fidelity is given by $F(M) = \frac{2M+1}{3M}$ [7], is preserved under stabilizer encoding for $M > 2$.
- To examine asymmetric cloning of logical qubits — where the two output copies are permitted different fidelities — and to determine whether stabilizer structure deforms the known physical-qubit fidelity trade-off curve.
- To analyse whether structural properties of stabilizer codes — such as code distance, degeneracy, and logical operator structure — affect achievable cloning fidelity.

- To establish either (i) an encoding-invariance theorem demonstrating that optimal cloning fidelity is preserved under isometric embeddings, or (ii) explicit deviations indicating that encoding imposes nontrivial constraints or enhancements on cloning performance.
- To identify connections between the encoded cloning problem and related contemporary topics, including quantum channel cloning [4] and encrypted cloning [3, 12], situating the present results within the broader landscape of cloning research.
- To support analytical results with numerical investigations for small stabilizer codes using semidefinite programming techniques.

5 Methodology

The proposed research will be conducted within the framework of quantum channel theory and stabilizer formalism, combining analytical and numerical techniques to study approximate cloning of encoded quantum states.

5.1 Mathematical Framework

Logical qubits will be modeled as states in a Hilbert space $\mathcal{H}_L$, embedded into a physical Hilbert space $\mathcal{H}_P$ via an isometric encoding map $V : \mathcal{H}_L \to \mathcal{H}_P$. The image of this map defines a stabilizer code subspace $\mathcal{C} \subseteq \mathcal{H}_P$.

Approximate cloning operations will be represented as completely positive trace-preserving (CPTP) maps

$$\mathcal{E} : \mathcal{B}(\mathcal{H}_P) \to \mathcal{B}(\mathcal{H}_P \otimes \mathcal{H}_P),$$

with inputs restricted to states supported on the code subspace $\mathcal{C}$.

Cloning performance will be quantified using average state fidelity over the logical Bloch sphere, as defined in Section 2.

5.2 Analytical Approach

The analytical component will focus on characterizing the optimal cloning fidelity as a constrained optimization problem over CPTP maps. Specifically, the study will:

- Analyze whether the optimization over channels restricted to encoded inputs is equivalent to the unconstrained optimization over the full Hilbert space.
- Employ symmetry and covariance arguments, analogous to those used in universal quantum cloning [2], to identify candidate optimal channels.
- Investigate channel equivalence under isometric embeddings to determine whether logical cloning reduces to physical cloning through structural transformations.
- Examine the role of stabilizer structure, including logical operators and code distance, in constraining admissible channel actions.

5.3 Numerical Methods

Where analytical solutions are intractable, semidefinite programming (SDP) techniques will be employed to compute upper bounds on achievable cloning fidelity for small stabilizer codes. The optimization over CPTP maps will be expressed as an SDP using standard Choi matrix representations of quantum channels.

These numerical investigations will serve to:

- Provide evidence for or against encoding invariance in specific code instances,
- Identify patterns that inform conjectures and guide analytical proofs,
- Validate theoretical bounds derived in the analytical component.

5.4 Scope and Research Roadmap

The primary investigation targets $1 \to 2$ symmetric universal cloning of a single logical qubit under unconstrained CPTP maps, with fidelity evaluated via the SDP formulation of Section 3.5. The framework is structured along three natural axes of extension, each representing a concrete research direction that can be pursued as the primary question resolves.

Axis 1 — Cloning regime. Beyond the $1 \to 2$ case, the SDP formulation extends directly to $1 \to M$ cloning by enlarging the output space to $\mathcal{H}_P^{\otimes M}$, allowing investigation of whether the physical-qubit scaling $F(M) = \frac{2M+1}{3M}$ [7] is preserved under encoding. Asymmetric cloning — where the two output copies carry different fidelities — can be addressed by introducing an output-weight parameter into the objective matrix M_V of the SDP, requiring only a minor modification to the numerical framework.

Axis 2 — Nearby topics. The CJ isomorphism framework developed here extends naturally to the cloning of quantum channels rather than states, following the superchannel formalism of Sekatski et al. [4]. Additionally, the Pauli-based structure of stabilizer codes is directly related to the encryption operators used in encrypted cloning [3], suggesting a structural connection between code subspace geometry and the encrypted cloning paradigm that may be investigated analytically.

Axis 3 — Code structure. Numerically, the SDP will be evaluated for a range of small stabilizer codes, beginning with the $[[3, 1, 1]]$ repetition code as a baseline and progressing to the $[[5, 1, 3]]$ perfect code and the $[[7, 1, 3]]$ Steane code. An additional constrained variant restricts $\mathcal{E}$ to transversally implementable channels by adding stabilizer-circuit constraints to the SDP, connecting the analysis to fault-tolerant quantum computation.

6 Expected Contributions

This thesis aims to provide a rigorous theoretical characterization of approximate quantum cloning in the presence of stabilizer encoding. The expected contributions are as follows:

- **Encoding-Invariance Theorem or Counterexample:** Establish either (i) a formal proof that optimal approximate cloning fidelity is invariant under isometric stabilizer encoding, or (ii) a demonstration that stabilizer structure induces measurable deviations from, or improvements upon, the universal cloning bound of 5/6, as motivated by Conjecture 1 of Section 2.1.

- **Channel-Theoretic Formulation of Logical Cloning:** Develop a precise SDP framework for approximate cloning of logical qubits as an optimization over CPTP maps restricted to encoded subspaces, extending standard cloning theory to the encoded setting and providing a reusable tool for subsequent investigations.

- **Characterization of Structural Effects:** Identify whether features such as code distance, degeneracy, and logical operator structure influence achievable cloning fidelity, including its $1 \to M$ scaling and asymmetric trade-off curve.

- **Analytical and Numerical Bounds:** Derive analytical results where possible and supplement them with SDP-based bounds for the $[[3, 1, 1]]$, $[[5, 1, 3]]$, and $[[7, 1, 3]]$ stabilizer codes, offering concrete evidence for or against encoding-induced modification of the fidelity.

- **Applications to Quantum Technologies:** Characterize the implications of the logical cloning bound for quantum cloud storage, quantum key distribution security thresholds, and quantum repeater fidelity, connecting the theoretical results to current experimental platforms [3, 4].

The results are expected to be of independent theoretical interest and to provide a basis for further research in quantum error correction, quantum channel theory, and fault-tolerant quantum computation.

7 Feasibility and Risk Analysis

The proposed research is theoretical in nature and builds upon well-established frameworks in quantum channel theory and stabilizer formalism, ensuring a solid methodological foundation. The problem is precisely defined and can be approached through a combination of analytical techniques and numerical optimization, making it suitable for completion within a standard Master's thesis timeframe.

A primary intellectual risk is that encoding invariance of approximate cloning fidelity may follow as a direct consequence of known structural properties of quantum channels, such as invariance under isometric embeddings. In this case, the problem may reduce to establishing equivalence between logical and physical cloning formulations.

However, even under this scenario, deriving a rigorous and explicit proof within the stabilizer framework would constitute a meaningful theoretical contribution, as such results are not currently formalized in the literature.

To mitigate this risk, the research will begin with explicit analysis of small stabilizer codes using semidefinite programming methods. These investigations will help identify structural patterns and guide the development of general analytical results. If nontrivial deviations from known cloning bounds are observed, the work will focus on characterizing the dependence of fidelity on code-specific parameters.

Overall, the project is considered feasible due to its clear formulation, availability of established mathematical tools, and the presence of multiple viable research directions that ensure a substantive outcome regardless of the final result [11].

8 Timeline

The proposed research is structured over a six-month period, with clearly defined phases balancing theoretical development and supporting numerical analysis.

- **Month 1:** Comprehensive literature review on approximate quantum cloning and stabilizer codes; formalization of the problem within the framework of quantum channel theory.
- **Month 2:** Development of mathematical framework; precise formulation of cloning fidelity for encoded states; initial exploration of channel representations and constraints.
- **Month 3:** Analytical investigation of small stabilizer codes; identification of structural patterns; formulation of conjectures regarding encoding invariance.
- **Month 4:** Development of general analytical results; attempt to prove encoding invariance or identify counterexamples; refinement of theoretical framework.
- **Month 5:** Numerical verification using semidefinite programming (SDP) techniques; computation of fidelity bounds for selected stabilizer codes; validation of analytical findings.
- **Month 6:** Consolidation of results; preparation of formal proofs; writing and revision of thesis document; final presentation preparation.

This timeline ensures steady progress from foundational understanding to rigorous analysis and final documentation within the available timeframe.

9 Alternative Research Directions

If encoding invariance of approximate cloning fidelity is established early in the investigation, or if the primary results are obtained ahead of schedule, the research will pivot to one or more of the following directions. Each constitutes a self-contained and technically substantive contribution.

9.1 Structural Equivalence of Logical and Physical Cloning Channels

The research will examine whether any optimal approximate cloning operation on encoded logical states induces an equivalent optimal cloning channel on the underlying physical Hilbert space, and vice versa. Establishing this equivalence formally would extend the intuition that isometric embeddings preserve operational capabilities to the level of CPTP maps, and would provide a channel-theoretic characterisation of encoding invariance beyond the fidelity scalar.

9.2 Cloning Under Fault-Tolerant Operational Constraints

The framework will be extended to constrained cloning scenarios in which the allowed operations are restricted to transversally implementable or stabilizer-circuit channels. In this setting, encoding may introduce nontrivial operational limitations that lower the achievable fidelity below the unconstrained bound, providing a rigorous analysis of how fault-tolerance requirements interact with the fundamental limits of state replication.

9.3 Encrypted Cloning and Stabilizer Code Structure

The Pauli-based encryption operators used in the encrypted cloning protocol of Yamaguchi and Kempf [3] share algebraic structure with stabilizer code generators. This direction will investigate whether stabilizer encoding enables or constrains encrypted cloning, and whether the code distance and logical operator structure determine which sets of encrypted clones can be jointly decrypted [13, 14]. This connects the present work to the rapidly developing field of uncloneable quantum cryptography [16].

9.4 Cloning Fidelity Under Realistic Noise Models

The analysis will be extended to examine whether stabilizer encoding alters the optimal cloning fidelity in the presence of realistic error models, such as depolarising or dephasing noise acting on the physical qubits. This direction connects the present work more directly with quantum error correction, since the code's ability to suppress noise may interact nontrivially with the channel's ability to clone, potentially revealing a trade-off between error protection and cloning performance.

These directions ensure that, regardless of the outcome of the primary investigation, the thesis will yield a meaningful and technically substantive contribution to the theory of quantum information processing.

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